8.4 Factoring Nonmonic Quadratics

The quadratic formula gives you the solutions to any quadratic equation. With a little work, you can use these solutions to factor any quadratic polynomial.

What can you do if the quadratic is not monic? A good mathematical habit is to make use of something you already know how to do to solve problems you are unsure about.
Minds in Action  episode 37

Tony and Sasha are trying to factor the quadratic $4x^2 + 36x + 45$.

Tony  It's not monic. Do we have to play with all the combinations?
Sasha  We could. Wait, I see something. $4x^2$ is the same as $(2x)^2$. So we could write the equation using $2x$ chunks.

$$(2x)^2 + 18(2x) + 45$$

Tony  Sure, you can do that, but it's still not monic.
Sasha  Well, no. But suppose I think of the $2x$ as one thing.

Sasha covers the first $2x$ with her left hand and the second $2x$ with her right hand.

Sasha  Do you see? It's something squared plus 18 times that something plus 45. Here, I'll change what's under my hand, the $2x$, to $z$. Now it looks better.

$$(z)^2 + 18z + 45$$

Tony  Cool! I can factor that by finding numbers that add to 18 and multiply to 45. So, 15 and 3 will work. Look at what I get.

$$z^2 + 18z + 45 = (z + 15)(z + 3)$$

Sasha  Remember, we used $z$ as a placeholder for $2x$, so now put the $2x$ back.

$$(z + 15)(z + 3) = (2x + 15)(2x + 3)$$

Tony  We should check by multiplying it out, just to be sure.
For You to Do

1. Expand \((2x + 15)(2x + 3)\) and make sure you get the original quadratic from episode 37.

Answers

For You to Do

1. \(4x^2 + 36x + 45\)
Simplify complicated problems. Sasha's idea of lumping a part of an expression into one unit is very useful in mathematics. People often say that \((2x)^2 + 18(2x) + 45\) is a monic quadratic in \(2x\). This means that if you think of the variable as \(2x\) instead of \(x\), the quadratic looks monic.
Minds in Action  
episode 38

Sasha and Tony want to try their method on other quadratics.

Tony  
Here’s one.

$$6x^2 + 11x - 10$$

I know the answer, because I got it by multiplying out two binomials. But I don’t think our method will work here.

Sasha  
We were lucky with $$4x^2 + 36x + 45$$ because of the leading 4. It’s a perfect square. So I could write the first term as $$(2x)^2$$. And I could write the second term as a multiple of $$2x$$ to the first power. So it all worked out.

Tony  
If it were an equation, we could multiply both sides by a number and maybe fix things up. But it’s a polynomial, not an equation.

Sasha and Tony sit silently for a while.

Sasha  
Well, we can still multiply by 6. Just remember that we did it so we can undo it later.

Tony  
Okay, but let’s not forget. It doesn’t sound legal.

Sasha  
Well, like I said, as long as we undo it in end, it should be fine. So, take the quadratic $$6x^2 + 11x - 10$$ and multiply it by 6.

$$6(6x^2 + 11x - 10) = 36x^2 + 66x - 60 \text{ (6 times)}$$

Tony  
I see what you’re doing. That “6 times” reminds us that we multiplied by 6. Now we can write the polynomial just like we did the last one.

$$(6x)^2 + 11(6x) - 60$$
Now it is monic in $6x$. Let $z = 6x$ and we get this.

$$z^2 + 11z - 60$$

I know how to factor that.

$$(z + 15)(z - 4)$$

**Sasha**  Now unreplace $z$ with $6x$.

$$(6x + 15)(6x - 4)$$

Oh, wait. There are common factors in each binomial, a 3 in the first one and a 2 in the second one. That's the 6 that I multiplied by in the first place! I'll pull the 3 and the 2 out.

$$3(2x + 5) \cdot 2(3x - 2) = 6(2x + 5)(3x - 2)$$

**Tony**  So now I can divide by 6 to undo your multiplying by 6.

$$(2x + 5)(3x - 2)$$

So $6x^2 + 11x - 10 = (2x + 5)(3x - 2)$, and we are done.

**Sasha**  This method will always work, so things just got a lot simpler. All we need to worry about now is factoring monics. We can do that by sums and products.
For You to Do

2. Factor $6x^2 - 31x + 35$ using Sasha's method.

3. Factor $6x^2 - 31x + 35$ using the quadratic formula.


For You to Do

2. $(3x - 5)(2x - 7)$
3. $(3x - 5)(2x - 7)$
4. Answers may vary depending on students’ preferences.

\[
(6x)^2 - 31(6x) + 210
\]

\[
(z^2 - 31z + 210)
\]

\[
(z-10)(z-21)
\]

\[
(6x-10)(6x-21)
\]

\[
2(3x-5)(2x-7)
\]

\[
(x(3x-5))(2x-7)
\]

\[
(3x-5)(2x-7)
\]
For Discussion

5. State Sasha and Tony’s method as an algorithm, a sequence of steps that describes exactly what to do.
Check Your Understanding

For Exercises 1 and 2, factor each polynomial.

1. a. \(9x^2 + 18x - 7\)  
   b. \(6x^2 - 31x + 35\)  
   c. \(15x^2 + 16x - 7\)  
   d. \(9x^2 + 62x - 7\)  
   e. \(9x^4 + 62x^2 - 7\)

2. a. \(9x^2 + 18xy - 7y^2\)  
   b. \(6x^2 - 31xy + 35y^2\)  
   c. \(15x^2 + 16xa - 7a^2\)  
   d. \(9x^2 + 62xb - 7b^2\)

3. When applying the scaling method to \(6x^2 - 31x + 35\), you follow these steps.
   - Look at the quadratic equation \(6x^2 - 31x + 35 = 0\).
   - Multiply both sides by 6.
   - Get a monic quadratic in \(6x\) on the left side.
   - Let \(z = 6x\) and work with the resulting quadratic in \(z\).

Describe how you could get the quadratic in \(z\) without the middle steps.

\[z^2 - 31z + A = 0\]
\[ a \cdot \begin{bmatrix} \frac{9}{4} x^4 + 62 x^2 - 7 \\ \frac{1}{2} x^2 + 62 x - 63 \\ 2 x^2 + 62 x - 63 \\ \frac{(z + 63)(2-1)}{(z + 63)(q x^2 - 1)} \\ (q x^2 + 63)(q x^2 - 1) \\ (x^2 + 7)(q x^2 - 1) \\ (x^2 + 7)(3x + 1)(3x - 1) \end{bmatrix} \]

\[ z = q x^2 \]
\[
\frac{9}{2} \left[ 9x^2 + 62x - 7 \right]
\]

\[9x = z\]

\[z = 9x\]

\[\begin{align*}
(9x)^2 + 62(9x) - 63 \\
(z + 63)(z - 1)
\end{align*}\]

\[\frac{9}{2} \cdot (X+7)(9x-1)\]

\[(X+7)(9x-1)\]
\[
15 \cdot \left[ \left( \frac{15}{x} \right)^2 + 16 \cdot \frac{15}{x} - 105 \right] = 0
\]

\[
(2 \cdot 3)^2 = (2 \cdot 3)^2
\]

\[
2 = 15x
\]

\[
2^2 + 16 \cdot 2 - 105
\]

\[
(2 + 21)(2 - 5)
\]

\[
2 = 15x \left( \frac{15x + 21}{15x - 5} \right)
\]

\[
3(5x+7) \cdot 5(3x-1)
\]

\[
15 \left( \frac{5x+7}{5x-1} \right) \left( 3x-1 \right)
\]

\[
(5x+7)(3x-1)
\]
\( Z = 3x \)
\( Z^2 + 6Z \rightarrow \)
\( (Z + 2)(Z - 1) \)
\( Z = 3x \)
\( (3x + 2)(3x - 1) \)
\( 9x^2 + 18x + y - 7y^2 \)
Answers

Exercises

1. a. \((3x + 7)(3x - 1)\)
   b. \((3x - 5)(2x - 7)\)
   c. \((5x + 7)(3x - 1)\)
   d. \((9x - 1)(x + 7)\)
   e. \((9x^2 - 1)(x^2 + 7)\) or 
      \((3x + 1)(3x - 1)(x^2 + 7)\)

2. a. \((3x + 7y)(3x - y)\)
   b. \((2x - 7y)(3x - 5y)\)
   c. \((3x - a)(5x + 7a)\)
   d. \((9x - b)(x + 7b)\)

3. Answers will vary. One observation is that the new monic polynomial in \(z\) will have as its constant term the product of the outer terms.
p698  #4-12
For Exercises 4-6, factor each polynomial.

4. a. \(-18x^2 - 65x - 7\)
   c. \(25 - 4x^2\)
   b. \(-18x^2 + 61x + 7\)
   d. \(18x^3 - 61x^2 - 7x\)

5. a. \(-18x^2 - 65x@ - 7\)
   c. \(25y^2 - 4x^2\)
   b. \(-18x^2 + 61xy + 7y^2\)
   d. \(18x^3 - 61x^2y - 7xy^2\)

6. a. \(4x^2 - 13x + 3\)
   c. \(4x^2 + 4x - 3\)
   e. \(4x^4 - 13x^2 + 3\)
   g. \(4(x - 1)^2 - 13(x - 1)^6 + 3\)
   b. \(4x^2 - 8x + 3\)
   d. \(4(x + 1)^2 + 4(x + 1) - 3\)
   f. \(4(x - 1)^4 - 13(x - 1)^2 + 3\)
   h. \((x^2 + 1)^2 - x^2\)

\[
\begin{align*}
4x^2 - 13x + 3 &= (4x - 3)(x - 1) \\
(4x)^2 - 13(4x) + 12 &= (4x - 3)(x - 1) \\
4x &= 3 \\
(2 - 12)(2 - 1) &= (4x - 12)(4x - 1) \\
4(x - 3)(4x - 1) &= 4(x - 3)(4x - 1) \\
(x - 3)(4x - 1) &= (x - 3)(4x - 1)
\end{align*}
\]
For Exercises 4–6, factor each polynomial.

4. a. $-18x^2 - 65x - 7$
   c. $25 - 4x^2$

5. a. $-18x^2 - 65xa - 7a^2$
   c. $25y^2 - 4x^2$

6. a. $4x^2 - 13x + 3$
   c. $4x^2 + 4x - 3$
   e. $4x^4 - 13x^2 + 3$
   g. $4(x - 1)^{12} - 13(x - 1)^6 + 3$

b. $-18x^2 + 61x + 7$
   d. $18x^3 - 61x^2 - 7x$
   b. $-18x^2 + 61xy + 7y^2$
   d. $18x^3 - 61x^2y - 7xy^2$

b. $4x^2 - 8x + 3$
   d. $4(x + 1)^2 + 4(x + 1) - 3$
   f. $4(x - 1)^4 - 13(x - 1)^2 + 3$
   h. $(x^2 + 1)^2 - x^2$

$$K = (x-1)^6$$
$$4K^2 - 13K + 3$$

$$(K-3)(4K-1)$$
$$(x-1)^6 - 3(4(x-1)^6 - 1)$$
\[\frac{4}{x^4} - 13x^2 + 3\]
\[(4x^2)^2 - 13(4x^2) + 12\]
\[z^2 - 13z + 12\]
\[(z-1)(z-12)\]
\[(4x^2-1)(4x^2-12)\]
\[1 \times 4(x^2-3) / 4\]
\[(2x-1)(2x+1)(x^2-3)\]
$$(-1) \left( (18x^2 + 65x + 7) \right) \left( (18x)^2 + 65(18x) + 126 \right)$$
$$= \left( z^2 + 65z + 126 \right)$$
$$= -1 \left( z + 2 \right) \left( z + 63 \right)$$
$$= -1 \left( 18x + 2 \right) \left( 18x + 63 \right)$$
$$= (-1)2(9x+1) \cdot 9(2x+7)$$
$$\frac{18}{-6(9x+1)(2x+7)}$$

$$5A \cdot (-1) \left( 9x + A \right) \left( 2x + 7A \right)$$
4. a. $-(9x + 1)(2x + 7)$
b. $-(9x + 1)(2x - 7)$
c. $(5 - 2x)(5 + 2x)$
d. $x(9x + 1)(2x - 7)$

5. a. $-(9x + a)(2x + 7a)$
b. $-(9x + y)(2x - 7y)$
c. $(5y + 2x)(5y - 2x)$
d. $x(9x + y)(2x - 7y)$
7. **Standardized Test Prep**  For what values of \( x \) is the equation \( 2x - \frac{15}{x} = 1 \) true?

A. 3  \hspace{1cm} B. \(-3, \frac{5}{2}\)  \hspace{1cm} C. \(\frac{5}{2}, 3\)  \hspace{1cm} D. \(-\frac{5}{2}, 3\)

8. Solve the equation \( 2x - \frac{3}{x} = 5 \) for \( x \).

9. For the equation \( x^2 - 6x + 7 = 0 \), let \( z = x - 3 \).
   a. Express the equation in terms of \( z \).
   b. Solve the equation in \( z \).
   c. Use your result from part (b) to solve the original equation.

\[
\begin{align*}
X^2 - 6X + 9 - 9 + 7 & = (X - 3)^2 - 2 \quad \Rightarrow \quad \frac{(-2)^2 - 2}{(z + 2)(z - 2)} \\
(X - 3)^2 & = \frac{(-2)^2 - 2}{(z + 2)(z - 2)} \quad \Rightarrow \quad X^2 = \frac{2}{(z + 2)(z - 2)}
\end{align*}
\]

10. For parts (a) and (b), solve the quadratic equation.
   a. \( 3x^2 + 8x - 35 = 0 \)
   b. \( x^2 + 8x - 105 = 0 \)
   c. Find a relationship between the solutions to these two equations.
7. D
8. \( x = 3 \) or \( x = -\frac{1}{2} \)
9. a. \( z^2 - 2 = 0 \)
   b. \( z = \pm \sqrt{2} \)
   c. \( x = 3 \pm \sqrt{2} \)
10. a. \( x = -5 \) or \( x = \frac{7}{3} \)
    b. \( x = -15 \) or \( x = 7 \)
    c. The solutions to the first equation are \( \frac{1}{3} \) times the solutions to the second equation.
11. Solve each pair of equations and compare the roots.
   a. \( x^2 - 8x + 7 = 0 \) and \( x^2 - 24x + 63 = 0 \)
   b. \( 2x^2 + 11x - 21 = 0 \) and \( 2x^2 + 22x - 84 = 0 \)
   c. \( 2x^2 + 11x - 21 = 0 \) and \( 2x^2 + 33x - 189 = 0 \)
   d. \( 2x^2 + 11x - 21 = 0 \) and \( 2x^2 + 55x - 525 = 0 \)
   e. \( 2x^2 + 11x - 21 = 0 \) and \( x^2 + 11x - 42 = 0 \)
   f. \( 3x^2 + 16x - 35 = 0 \) and \( x^2 + 16x - 105 = 0 \)
   g. \( 3x^2 + 16x - 32 = 0 \) and \( x^2 + 16x - 96 = 0 \)

12. Find an equation with roots that are as follows.
   a. 7 times the roots of \( x^2 - 8x + 7 = 0 \)
   b. 7 times the roots of \( 2x^2 + 11x - 21 = 0 \)
   c. 2 times the roots of \( 2x^2 + 11x - 21 = 0 \)
   d. 3 times the roots of \( 3x^2 + 11x - 21 = 0 \)
   e. 5 times the roots of \( 5x^2 + 11x - 21 = 0 \)
Lesson 8.4  

Exercises

6a. \((4x - 1)(x - 3)\)
b. \((2x - 3)(2x - 1)\)
c. \((2x + 3)(2x - 1)\)
d. \((2x + 5)(2x + 1)\)
e. \((4x^2 - 1)(x^2 - 3)\) or 
\((2x + 1)(2x - 1)(x^2 - 3)\)
f. \((4(x^2 - 1)^2 - 1)((x^2 - 1) - 3)\)
or \((x^2 - 2x - 2)(2x - 1)(2x - 3)\)
g. \((4(x - 1)^6 - 1)((x - 1)^6 - 3)\)
or \((2(x - 1)^3 - 1)(2(x - 1)^3 + 1)\)
\((x - 1)^6 - 3)\)
h. \((x^2 + 1 - x)(x^2 + 1 + x)\) or 
\((x^2 - x + 1)(x^2 - x + 1)\)

11a. \(x = 1\) and \(x = 7\); \(x = 3\) and 
\(x = 21\). The solutions to the second 
equation are three times the solutions to the first equation. b. \(x = -7\) and
\(x = -\frac{3}{2}\); \(x = -14\) and \(x = 3\). The 
solutions to the second equation are 
two times as great. c. \(x = -7\) and
\(x = -\frac{3}{2}\); \(x = -21\) and \(x = -\frac{9}{2}\). The 
solutions to the second equation are 
three times as great. d. \(x = -7\) and
\(x = -\frac{3}{2}\); \(x = -35\) and \(x = -\frac{15}{2}\). The 
solutions to the second equation are 
five times as great. e. \(x = -7\) and
\(x = -\frac{3}{2}\); \(x = -14\) and \(x = 3\). The 
solutions to the second equation are 
two times as great. f. \(x = -7\) and
\(x = -\frac{5}{3}\); \(x = -21\) and \(x = 5\). The 
solutions to the second equation are 
three times as great. g. \(x \approx -6.883\)
and \(x \approx 1.550\); \(x \approx -20.649\) and
12. a–e. Answers may vary. Samples are given.

a. \( x^2 - 56x + 343 = 0 \)
b. \( 2x^2 + 77x - 1029 = 0 \)
c. \( x^2 + 11x - 42 = 0 \)
d. \( x^2 + 11x - 63 = 0 \)
e. \( x^2 + 11x - 105 = 0 \)
p700 #1-8

HW p701 #1-13